

On Pre- $\hat{\delta}$ -Separation Axioms in Ideal Topological Spaces

Muna Jabbar Mohammad

Faik Jameel H. Mayah*

Department of Mathematics, College of Education for Pure Sciences, Wasit University

Department of Physics, College of Sciences, Wasit University, Iraq*

E-mail: Jbarmny7@gmail.comfaik.mayah@gmail.com*

Abstract

The separation axioms in particular that concerned with ideal topological spaces are discussed widely in the literature because they classify the spaces depending on the validity of each axiom of separating either the points or the subsets. In this work, new type of separation axioms in ideal topological spaces are defined which is more natural than these appear in the literature. This new classis of separation axioms depends on the notion of pre- $\hat{\delta}$ -open. Properties of these sorts are studied and the relationships among these concepts are discussed.

Keywords: Pre- $\hat{\delta}$ -closed, Pre- $\hat{\delta}$ -open, $\hat{\delta}$ - \mathbb{T}_i -spaces, $p\hat{\delta}$ - \mathbb{T}_i -spaces.

1. Introduction

A nonempty collection \mathbb{I} of subsets in a topological-space (X, \mathbb{T}) is said to be an ideal if it satisfies

- $\mathcal{A} \in \mathbb{I}$ and $\mathcal{B} \subseteq \mathcal{A}$ implies $\mathcal{B} \in \mathbb{I}$.
- $\mathcal{A} \in \mathbb{I}$ and $\mathcal{B} \in \mathbb{I}$ implies $\mathcal{A} \cup \mathcal{B} \in \mathbb{I}$.

A topological-space (X, \mathbb{T}) with an ideal \mathbb{I} is called an ideal topological-space or simply ideal space. If $P(X)$ is the set of all subsets of X , a set operator $(\cdot)^*: P(X) \rightarrow P(X)$ is called a local function [1] of a subset \mathcal{A} with respect to the topology \mathbb{T} and ideal \mathbb{I} , which is defined as $A^*(X, \mathbb{T}) = \{x \in X : \mathcal{W} \cap \mathcal{A} \notin \mathbb{I}, \forall \mathcal{W} \in \mathbb{T}(X)\}$ where $\mathbb{T}(X) = \{\mathcal{W} \in \mathbb{T} : x \in \mathcal{W}\}$. A kuratowski closure operator $cl^*(\cdot)$ for a topology $\mathbb{T}^*(\mathbb{I}, \mathbb{T})$, called the $*$ -topology; finer than \mathbb{T} is defined by $cl^*(\mathcal{A}) = A^*(\mathbb{I}, \mathbb{T}) \cup \mathcal{A}$ [2]. Levine [3]; velicko [4] introduced the notions of generalized closed (briefly g -closed) and δ -closed sets respectively and studied their basic properties. The notion of $\mathbb{I}g$ -closed sets first introduced by Dontchev [5] in 1999;

Navaneetha Krishanan and Joseph [6] further investigated and characterized $\mathbb{I}g$ -closed sets. Julian Dontchev and Maximilian Ganster [7]; Yuksel; Acikgoz and Noiri [8] introduced and studied the notions of δ -generalized closed (briefly δg -closed) and δ - \mathbb{I} -closed sets respectively.

Pre- $\hat{\delta}$ -closed sets a novel type of sets that will be defined in this paper with fundamental characteristics.

In this paper we define some topological concepts called $\hat{\delta}$ - \mathbb{T}_i -space and pre- $\hat{\delta}$ - \mathbb{T}_i -space [for all $i = 0, 1, 2$] in the context of ideal topological spaces, and we study the relationship among these structures.

2. Fundamental Concepts

Definition 2.1. Let \mathcal{A} subset of a topological-space (X, \mathbb{I}) is said a:

- Semi-open set [9] if $\mathcal{A} \subseteq cl(int(\mathcal{A}))$.
- Semi-closed set [10] if $int(cl(\mathcal{A})) \subseteq \mathcal{A}$.
- Pre-open set [11] if $\mathcal{A} \subseteq int(cl(\mathcal{A}))$.

- Pre-closed set [12] if $cl(int(\mathcal{A})) \subseteq \mathcal{A}$.
- Regular open set [13] if $\mathcal{A} = int(cl(\mathcal{A}))$.
- Regular closed set [14] if $\mathcal{A} = cl(int(\mathcal{A}))$.

The semi- closure (respectively, pre-closure) of a subset \mathcal{A} of (\mathbb{X}, \mathbb{T}) is the intersection of all semi-closed (respectively, pre-closed) sets containing \mathcal{A} and is denoted by $scl(\mathcal{A})$ (respectively, $pcl(\mathcal{A})$).

Definition 2.2. [8] Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal topological-space, let \mathcal{A} a subset of \mathbb{X} and x is a point of \mathbb{X} . Then: x is called a δ - \mathbb{I} -cluster points of \mathcal{A} if $\mathcal{A} \cap (int\ cl^*(\mathcal{W})) \neq \emptyset$ for all open neighborhood \mathcal{W} of x .

The family of each δ - \mathbb{I} -cluster points of \mathcal{A} is said the δ - \mathbb{I} -closure of \mathcal{A} and is denoted by $[\mathcal{A}]_{\delta-\mathbb{I}}$. A subset \mathcal{A} is called to be δ - \mathbb{I} -closed if $[\mathcal{A}]_{\delta-\mathbb{I}} = \mathcal{A}$.

The complement of a δ - \mathbb{I} -closed set of \mathbb{X} is called to be δ - \mathbb{I} -open.

Remark 2.1. We can write $[\mathcal{A}]_{\delta-\mathbb{I}} = \{x \in \mathbb{X} : int(cl * (\mathcal{W}) \cap \mathcal{A} \neq \emptyset, \text{ for all } \mathcal{W} \in \mathbb{T}(\mathbb{X})\}$. We use the notation $\sigma cl(\mathcal{A}) = [\mathcal{A}]_{\delta-\mathbb{I}}$.

Lemma 2.1. [8] Let \mathcal{A} and \mathcal{B} be subset of an ideal topological-space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$. Then the following properties satisfy:

- $\mathcal{A} \subseteq \sigma cl(\mathcal{A})$.
- If $\mathcal{A} \subset \mathcal{B}$, then $\sigma cl(\mathcal{A}) \subset \sigma cl(\mathcal{B})$.
- $\sigma cl(\mathcal{A}) = \bigcap \{ \mathcal{G} \subset \mathbb{X} : \mathcal{A} \subset \mathcal{G} \text{ and } \mathcal{G} \text{ is } \delta\text{-}\mathbb{I}\text{-closed} \}$.
- If \mathcal{A} is δ - \mathbb{I} -closed set of \mathbb{X} for all $\alpha \in \Delta$; then $\bigcap \{ \mathcal{A}_\alpha : \alpha \in \Delta \}$ is δ - \mathbb{I} -closed.
- $\sigma cl(\mathcal{A})$ is δ - \mathbb{I} -closed.

Remark 2.2. It is well-Known that the family of regular open sets of (\mathbb{X}, \mathbb{T}) is a basis for a topology which is weaker than \mathbb{T} . This topology is called the semi-regularization of \mathbb{T} and is denoted by \mathbb{T}_s . Actually, \mathbb{T}_s is the same as the family of δ -open sets of (\mathbb{X}, \mathbb{T}) .

Lemma 2.2. [8] Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal topological-space, and $\mathbb{T}_{\delta-\mathbb{I}} = \{ \mathcal{A} \subset \mathbb{X} : \mathcal{A} \text{ is } \delta\text{-}\mathbb{I}\text{-open set of } (\mathbb{X}, \mathbb{T}, \mathbb{I}) \}$. Then $\mathbb{T}_{\delta-\mathbb{I}}$ is a topology such that $\mathbb{T}_s \subset \mathbb{T}_{\delta-\mathbb{I}} \subset \mathbb{T}$.

Remark 2.3. [8] \mathbb{T}_s (respectively, $\mathbb{T}_{\delta-\mathbb{I}}$) is the topology created by the family of δ -open sets (respectively, δ - \mathbb{I} -open sets).

Lemma 2.3. Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal topological space, and \mathcal{A} a subset of \mathbb{X} .

$\sigma cl(\mathcal{A}) = \{x \in \mathbb{X} : \mathcal{A} \cap int(cl^*(\mathcal{W})) \neq \emptyset; \text{ for each } \mathcal{W} \in \mathbb{T}(\mathbb{X})\}$ is closed.

Proof. If $x \in cl(\sigma cl(\mathcal{A}))$, and $\mathcal{W} \in \mathbb{T}(\mathbb{X})$, then $\mathcal{W} \cap \sigma cl(\mathcal{A}) \neq \emptyset$.

Then $y \in \mathcal{W} \cap \sigma cl(\mathcal{A})$ for some $y \in \mathbb{X}$.

Since $\mathcal{W} \in \mathbb{T}(y)$ and $y \in \sigma cl(\mathcal{A})$, from the definition of $\sigma cl(\mathcal{A})$ we have $\mathcal{A} \cap int(cl^*(\mathcal{W})) \neq \emptyset$. Therefore, $x \in \sigma cl(\mathcal{A})$. So $cl(\sigma cl(\mathcal{A})) \subset \sigma cl(\mathcal{A})$ and hence $\sigma cl(\mathcal{A})$ is closed.

Definition 2.3. Let (\mathbb{X}, \mathbb{T}) be a topological-space; a subset \mathcal{A} of \mathbb{X} is said to be :

- g -closed set [3] if $cl(\mathcal{A}) \subseteq \mathcal{W}$, whenever $\mathcal{A} \subset \mathcal{W}$ and \mathcal{W} is open in (\mathbb{X}, \mathbb{T})
- δ -closed set [4] if $\mathcal{A} = cl_\delta(\mathcal{A})$; where $\delta cl(\mathcal{A}) = cl_\delta(\mathcal{A}) = \{x \in \mathbb{X} : (int(cl(\mathcal{W})) \cap \mathcal{A}) \neq \emptyset, \mathcal{W} \in \mathbb{T} \text{ and } x \in \mathcal{W}\}$.
- δ -generalized closed set (short, δg -closed) set [7] if $cl_\delta(\mathcal{A}) \subseteq \mathcal{W}$, whenever $\mathcal{A} \subseteq \mathcal{W}$ and \mathcal{W} is open.
- $\delta \hat{g}$ -closed set [15] if $cl_\delta(\mathcal{A}) \subseteq \mathcal{W}$, whenever $\mathcal{A} \subseteq \mathcal{W}$ and \mathcal{W} is \hat{g} -open set in (\mathbb{X}, \mathbb{T}) .

Definition 2.4. Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal space. A subset \mathcal{A} of is said to be:

$\mathbb{I}g$ -closed set [5] if $\mathcal{A} * \subseteq \mathcal{W}$, whenever $\mathcal{A} \subseteq \mathcal{W}$ and \mathcal{W} is open in \mathbb{X} .

Definition 2.5. [16] Let $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ be an ideal space, a subset \mathcal{A} of is called $\hat{\delta}$ -closed if $\sigma cl(\mathcal{A}) \subset \mathcal{W}$, whenever $\mathcal{A} \subset \mathcal{W}$, and \mathcal{W} is open in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$.

The complement of $\hat{\delta}$ -closed set in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called $\hat{\delta}$ -open set in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$.

Definition 2.6. In ideal topological-space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$; let $\mathcal{A} \subset \mathbb{X}$, \mathcal{A} is called pre- $\hat{\delta}$ -closed if $\sigma cl(\mathcal{A}) \subset \mathcal{W}$ whenever $\mathcal{A} \subset \mathcal{W}$ and \mathcal{W} is pre-open in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$.

The complement of pre- $\hat{\delta}$ -closed in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called pre- $\hat{\delta}$ -open set in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$.

Example 2.1. Let $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}, \{e_2\}, \{e_1, e_2\}\}$, $\mathbb{I} = \{\phi, \{e_3\}\}$. Let $\mathcal{A} = \{e_1, e_3\}$ then \mathcal{A} is pre- $\hat{\delta}$ -closed.

Remark 2.4. Each Pre- $\hat{\delta}$ -closed is $\hat{\delta}$ -closed, but the opposite of is not true. It is clear from the following example.

Example 2.2. Let $\mathbb{X} = \{e_1, e_2, e_3, e_4\}$; $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}, \{e_2\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}$, and $\mathbb{I} = \{\phi, \{e_1\}\}$. Let $\mathcal{A} = \{e_1, e_4\}$, then \mathcal{A} is $\hat{\delta}$ -closed but not pre- $\hat{\delta}$ -closed.

Remark 2.5.

- The collection of all $\hat{\delta}$ -closed sets in (\mathbb{X}, \mathbb{I}) denoted by $\hat{\delta}\mathcal{C}(\mathbb{X})$, the family of all $\hat{\delta}$ -open sets in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ denoted by $\hat{\delta}\mathcal{O}(\mathbb{X})$.
- The collection of all pre- $\hat{\delta}$ -closed sets in $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ denoted by $P\hat{\delta}\mathcal{C}(\mathbb{X})$, the family of all pre- $\hat{\delta}$ -open sets denoted by $P\hat{\delta}\mathcal{O}(\mathbb{X})$.

3. Pre- $\hat{\delta}$ -Separation Axioms

This section is to present new classes of separation axioms by using the notion of pre- $\hat{\delta}$ -open. Properties of these sorts were studied and the relationships among these concepts were discussed.

Definition 3.1. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called $\hat{\delta}$ - \mathbb{T}_0 -space if for each pair of distinct points $x, y \in \mathbb{X}$ there exist an $\hat{\delta}$ -open set containing only one of them.

Example 3.1. Let $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}\}$ and $\mathbb{I} = \{\phi, \{e_1\}\}$. Then $\hat{\delta}\mathcal{C}(\mathbb{X}) = \{\mathbb{X}, \phi, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$

and $\hat{\delta}\mathcal{O}(\mathbb{X}) = \{\mathbb{X}, \phi, \{e_1, e_3\}, \{e_1, e_2\}, \{e_3\}, \{e_2\}, \{e_1\}\}$. Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}$ - \mathbb{T}_0 -space.

Definition 3.2. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called pre- $\hat{\delta}$ - \mathbb{T}_0 -space (briefly $p\hat{\delta}$ - \mathbb{T}_0 -space) if for each pair of distinct points $x, y \in \mathbb{X}$ there exist an pre- $\hat{\delta}$ -open set containing only one of them.

Example 3.2. In $(\mathbb{X}, \mathbb{T}, \mathbb{I})$, $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = \{\mathbb{X}, \phi, \{e_1\}, \{e_2\}, \{e_1, e_2\}\}$, $\mathbb{I} = \{\phi, \{e_3\}\}$, the family of pre-open subsets of \mathbb{X} is $PO(\mathbb{X}) = \{\mathbb{X}, \phi, \{e_1\}, \{e_1, e_2\}, \{e_2\}\}$. And $\hat{\delta}\mathcal{C}(\mathbb{X}) = \{\mathbb{X}, \phi, \{e_3\}, \{e_1, e_3\}, \{e_2, e_3\}\}$. Also, $P\hat{\delta}\mathcal{O}(\mathbb{X}) = \{\mathbb{X}, \phi, \{e_1, e_2\}, \{e_2\}, \{e_1\}\}$. Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_0 -space.

Theorem 3.1. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_0 -space if and only if for each elements $x \neq y$ there is a pre- $\hat{\delta}$ -closed set containing only one of them.

Proof. Let x and y are two distinct elements in \mathbb{X} . Since \mathbb{X} is a $p\hat{\delta}$ - \mathbb{T}_0 -space, then there is a pre- $\hat{\delta}$ -open set \mathcal{U} containing only one of them, then $\setminus \mathcal{U}$ is a pre- $\hat{\delta}$ -closed set containing the other one.

Conversely. Let x and y are two distinct elements in \mathbb{X} . And there is a pre- $\hat{\delta}$ -closed set \mathcal{V} containing only one of them. Then $\setminus \mathcal{V}$ is a pre- $\hat{\delta}$ -open set containing the other one.

Proposition 3.1. If $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_0 -space then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}$ - \mathbb{T}_0 -space.

Proof. Let x and y are two distinct elements in \mathbb{X} . Since $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_0 -space, then there is a pre- $\hat{\delta}$ -open set \mathcal{U} containing only one of them. Since every pre- $\hat{\delta}$ -open set is a $\hat{\delta}$ -open set, then \mathcal{U} is a $\hat{\delta}$ -open set. Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}$ - \mathbb{T}_0 -space.

Definition 3.3. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called pre- $\hat{\delta}$ - \mathbb{T}_1 -space (briefly $p\hat{\delta}$ - \mathbb{T}_1 -

space) if for each elements $x \neq y$, there is pre- $\hat{\delta}$ -open sets $\mathcal{U}_1, \mathcal{U}_2$, satisfies $x \in \mathcal{U}_1 \setminus \mathcal{U}_2$ and $y \in \mathcal{U}_2 \setminus \mathcal{U}_1$.

When $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space, it will lead to that $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Example 3.3. Let $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = P(\mathbb{X})$ and $\mathbb{I} = \emptyset$, then $\hat{\delta}O(\mathbb{X}) = \hat{\delta}C(\mathbb{X}) = P\hat{\delta}O(\mathbb{X}) = P\hat{\delta}C(\mathbb{X}) = P(\mathbb{X})$. Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Proposition 3.2. If $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Proof. Let x and y are two distinct elements in \mathbb{X} . Since $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space, then there is pre- $\hat{\delta}$ -open sets \mathcal{U}, \mathcal{V} such that $x \in \mathcal{V}$ and $y \in \mathcal{V} \setminus \mathcal{U}$. Since every pre- $\hat{\delta}$ -open set is a $\hat{\delta}$ -open set. Then, \mathcal{U} and \mathcal{V} are a $\hat{\delta}$ -open sets. Then (\mathbb{X}, \mathbb{I}) is a $\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Proposition 3.3. If \mathbb{X} is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space, implies that $p\hat{\delta}\text{-}\mathbb{T}_0$ -space.

Proof. Let x and y are two distinct elements in \mathbb{X} . Since $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space, then there exists pre- $\hat{\delta}$ -open sets \mathcal{U}, \mathcal{V} , whenever $x \in \mathcal{V}, y \in \mathcal{V} \setminus \mathcal{U}$. Then there is a pre- $\hat{\delta}$ -open set \mathcal{U} containing only one of them. Then, \mathbb{X} is a $p\hat{\delta}\text{-}\mathbb{T}_0$ -space.

Remark 3.1. The inverse meaning implied in Proposition (3.3) does not true, in general, the following example explains this fact.

Example 3.4. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_0$ -space, where $\mathbb{X} = \{e_1, e_2, e_3\}$, $\mathbb{T} = \{\mathbb{X}, \emptyset, \{e_1, e_2\}, \{e_1\}, \{e_2\}, \{e_2, e_3\}\}$ and $\mathbb{I} = \{\emptyset\}$. Then, $PO(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{e_1\}, \{e_1, e_2\}, \{e_2, e_3\}, \{e_2\}\}$. And, $P\hat{\delta}C(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{e_3\}, \{e_2, e_3\}, \{e_1\}, \{e_1, e_3\}\}$. Also, $P\hat{\delta}O(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{e_1, e_2\}, \{e_1\}, \{e_2\}, \{e_2, e_3\}\}$. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is not $p\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Since for the elements $e_3 \neq e_2$ there is no pre- $\hat{\delta}$ -open set \mathcal{U} containing e_3 which does not contain e_2 .

Theorem 3.2. For any ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$, it is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space if and only if for all elements $x \neq y$, there exists two pre- $\hat{\delta}$ -closed sets \mathcal{W}_1 and \mathcal{W}_2 such that $x \in \mathcal{W}_1 \setminus \mathcal{W}_2$ and $y \in \mathcal{W}_2 \setminus \mathcal{W}_1$

Proof. Let x and y be two distinct elements in \mathbb{X} . Since \mathbb{X} is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space, then there is two pre- $\hat{\delta}$ -open sets \mathcal{U}, \mathcal{V} , such that $x \in \mathcal{U} \setminus \mathcal{V}, y \in \mathcal{V} \setminus \mathcal{U}$. Then there exists pre- $\hat{\delta}$ -closed sets $\mathbb{X} \setminus \mathcal{U}$ and $\mathbb{X} \setminus \mathcal{V}$, such that $x \in \mathbb{X} \setminus \mathcal{V} - \mathbb{X} \setminus \mathcal{U}, y \in \mathbb{X} \setminus \mathcal{U} - \mathbb{X} \setminus \mathcal{V}$, where $\mathcal{W}_1 = \mathbb{X} \setminus \mathcal{V}$ and $\mathcal{W}_2 = \mathbb{X} \setminus \mathcal{U}$.

Then there exists two pre- $\hat{\delta}$ -closed sets \mathcal{W}_1 and \mathcal{W}_2 satisfy, $x \in (\mathcal{W}_1 \cap \mathcal{W}_2^c)$ and $y \in (\mathcal{W}_2 \cap \mathcal{W}_1^c)$, therefor $x \in (\mathcal{W}_1 - \mathcal{W}_2)$ and $y \in (\mathcal{W}_2 - \mathcal{W}_1)$.

Conversely. Let x and y be two distinct elements in \mathbb{X} , and there exists two pre- $\hat{\delta}$ -closed sets \mathcal{W}_1 and \mathcal{W}_2 satisfy, $x \in (\mathcal{W}_1 \cap \mathcal{W}_2^c)$ and $y \in (\mathcal{W}_2 \cap \mathcal{W}_1^c)$. Therefor $x \in (\mathcal{W}_1 - \mathcal{W}_2)$ and $y \in (\mathcal{W}_2 - \mathcal{W}_1)$. Then there exists pre- $\hat{\delta}$ -open set $(\mathbb{X} - \mathcal{W}_1)$ and $(\mathbb{X} - \mathcal{W}_2)$, whenever $x \in \mathbb{X} \setminus \mathcal{W}_2 - \mathbb{X} \setminus \mathcal{W}_1, y \in \mathbb{X} \setminus \mathcal{W}_1 - \mathbb{X} \setminus \mathcal{W}_2$, where $\mathcal{U} = \mathbb{X} \setminus \mathcal{W}_2$ and $\mathcal{V} = \mathbb{X} \setminus \mathcal{W}_1$.

Proposition 3.4. If $\{x\}$ is a pre- $\hat{\delta}$ -closed set for all $x \in \mathbb{X}$, then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Proof. Let x and y be two distinct elements in \mathbb{X} . $\{x\}, \{y\}$ are pre- $\hat{\delta}$ -closed sets. So $\mathbb{X} \setminus \{x\}$ and $\mathbb{X} \setminus \{y\}$ are pre- $\hat{\delta}$ -open sets. Then there exists pre- $\hat{\delta}$ -open sets \mathcal{U} and \mathcal{V} , where $\mathcal{U} = \mathbb{X} \setminus \{y\}$ and $\mathcal{V} = \mathbb{X} \setminus \{x\}$ such that $x \in \mathcal{U} \setminus \mathcal{V}$ and $y \in \mathcal{V} \setminus \mathcal{U}$. Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}\text{-}\mathbb{T}_1$ -space.

Definition 3.4. The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called $\hat{\delta}\text{-}\mathbb{T}_2$ -space if for all elements $x \neq y$ there is disjoint $\hat{\delta}$ -open sets \mathcal{U} and \mathcal{V} satisfies $x \in \mathcal{U}$ and $y \in \mathcal{V}$.

Example 3.5. Let $\mathbb{X} = \{e_1, e_2, e_3\}$ and $\mathbb{T} = \{\mathbb{X}, \emptyset, \{e_1\}\}$ be a topology on \mathbb{X} with an ideal

$\mathbb{I} = \{\emptyset, \{e_1\}\}$. Then $\hat{\delta}C(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{e_2\}, \{e_3\}, \{e_1, e_2\}, \{e_1, e_3\}, \{e_2, e_3\}\}$,
 $\hat{\delta}O(\mathbb{X}) = \{\mathbb{X}, \emptyset, \{e_1, e_3\}, \{e_1, e_2\}, \{e_3\}, \{e_2\}, \{e_1\}\}$.
 Then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is $\hat{\delta}$ - \mathbb{T}_2 -space.

Definition 3.5.

The ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is called pre- $\hat{\delta}$ - \mathbb{T}_2 -space (briefly $p\hat{\delta}$ - \mathbb{T}_2 -space) if for all element $x \neq y$ there is disjoint pre- $\hat{\delta}$ -open sets \mathcal{U} and \mathcal{V} satisfies $x \in \mathcal{U}$ and $y \in \mathcal{V}$.

Proposition 3.5. If $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_2 -space then $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}$ - \mathbb{T}_2 -space.

Proof. Let x and y are two distinct elements in \mathbb{X} . Since (\mathbb{X}, \mathbb{I}) is a $p\hat{\delta}$ - \mathbb{T}_2 -space, then there are two pre- $\hat{\delta}$ -open sets \mathcal{U} and \mathcal{V} satisfying $x \in \mathcal{U}, y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.
 Since every pre- $\hat{\delta}$ -open set is a $\hat{\delta}$ -open set. Then \mathcal{U} and \mathcal{V} are $\hat{\delta}$ -open set satisfy $x \in \mathcal{U}, y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$.

Proposition 3.6. If the ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_2 -space then it is a $p\hat{\delta}$ - \mathbb{T}_1 -space.

Proof. Let x and y be two distinct elements in \mathbb{X} . Since $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_2 -space, then there exists pre- $\hat{\delta}$ -open set \mathcal{U} and \mathcal{V} , satisfy $x \in \mathcal{U}, y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Then there exists pre- $\hat{\delta}$ -open set \mathcal{U} and \mathcal{V} , such that $x \in \mathcal{V}$ and $y \in \mathcal{V} \setminus \mathcal{U}$.

Remark 3.2. The inverse meaning implied in Proposition (3.6), does not true in general by the following example.

Example 3.6. Consider the ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ such that $\mathbb{X} = \mathbb{N}$, the set of all natural numbers, $\mathbb{T} = \mathbb{T}_{cof}$, the set of all complement finite topology and $\mathbb{I} = \{\emptyset\}$.
 Then, $PO(\mathbb{X}) = \{\mathcal{U} \subseteq \mathbb{N} \text{ is an infinite set}\} \cup \{\emptyset\}$, and $P\hat{\delta}O(\mathbb{X}) = \mathbb{T} = \mathbb{T}_{cof}$, $P\hat{\delta}C(\mathbb{X}) = \{\mathcal{U} \subseteq \mathbb{X}: \mathcal{U} \text{ is a finite set}\} \cup \mathbb{X}$. It is clear that (\mathbb{X}, \mathbb{I}) is a $p\hat{\delta}$ - \mathbb{T}_1 -space, but it is not $p\hat{\delta}$ - \mathbb{T}_2 -space.

Remark 3.3. From the previous properties, we can get that if $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $p\hat{\delta}$ - \mathbb{T}_i -space, $i = \{0, 1, 2\}$. Then the ideal topological space $(\mathbb{X}, \mathbb{T}, \mathbb{I})$ is a $\hat{\delta}$ - \mathbb{T}_i -space. But the inverse meaning implied may be void, as shown in the Diagram(1).

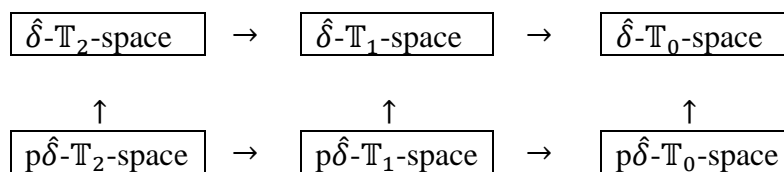


Figure1: Relationships among the $p\hat{\delta}$ - \mathbb{T}_i -spaces and $\hat{\delta}$ - \mathbb{T}_i -spaces.

4. Conclusions

New type of separation axioms are defined and discussed, it is called pre- $\hat{\delta}$ separation axioms. They are more natural than these appear in previous works. The relation-

ships between pre- $\hat{\delta}$ - \mathbb{T}_i spaces are found, see Figure 3.1. The properties that appear in this work hold in ideal topological space rather than topological space.

References

- 1- Kannan, K. *On $\hat{\beta}$ -Generalized closed sets and open sets in topological spaces*, Int. Journal of Math. Analysis, Vol. 6, No. 57, pp. 2819-2828, 2012.
- 2- Stone, M. *Application of the theory of Boolean rings to general topology*, Trans. Amer. Math. Soc., 41, pp. 374-481, 1937.
- 3- Levine, N. *Semi-open sets and semi-continuity in topological spaces*. Amer Math. Monthly, 70, pp. 36-41, 1963.
- 4- Veera Kumar, M.K.R.S. *\hat{g} -closed sets in topological spaces*, Bull. Allah. Math. Soc, 18, pp. 99-112, 2003.
- 5- Dontchev, J. and M. Ganster. *On δ -generalized closed sets and $T_{3/4}$ -spaces*, Mem. Fac. Sci. Kochi Univ. Ser. A, Math., 17, pp. 15-31, 1996.
- 6- Nagaveni, N. *Studies on generalizations of homeomorphisms in topological spaces*, Ph. D., Thesis, Bharathiar University. Coimbatore, 1999.
- 7- Bhattacharya, P. and B. K. Lahiri. *Semi-generalized closed sets in topology*, Indian J. Math., 29, pp. 375-382, 1987.
- 8- Velicko, N.V. *H-closed topological spaces*, Amer. Math. Soc. Transl., 78, pp. 103-118, 1968.
- 9- Lellis Thivagar, M, B. Meera Devi and E. Hatir. *$\delta\hat{g}$ -closed sets in topological spaces*, Gen. Math. Notes, Vol.1, No.2, pp. 17-25, 2010.
- 10- N. Levine. *Generalized closed sets in topology*, Rend. Del Circ. Mat. Di Palermo, vol. 19, no. 1, pp. 89-96, 1970.
- 11- Maki, H., J. Umehara, and K. Yamamura. *Characterizations of $T_{1/2}$ -spaces using generalized v -sets*, Indian Journal of pure and applied Mathematics, Vol. 19, No.7, pp. 634-640, 1988.
- 12- Jiling Cao, M. Ganster, C. Konstadilaki, Ivan L. Reilly. *On preclosed sets and their generalizations*, Houston Journal of Mathematics, 28 (4), pp. 771-780, 2002.
- 13- Ravi, O., S. Tharmar, M. Sangeetha, J. Antony Rex Rodrigo. **g-closed sets in ideal topological spaces*, Jordan Journal of Mathematics and statistics (JJMS) 6(1), pp.1-13, 2013.
- 14- Umit Karabiyik, Aynur Keskin Kaymakci. *Some New Regular Generalized Closed Sets in Ideal Topological Spaces*, European Journal of Pure and Applied Mathematics , Vol. 9, No. 4, 2016, 434-442, 2016.
- 15- Kuratowski, K. *Topology*, Vol.1, Academic Press, New York, NY, USA, 1966.
- 16- M. Navaneethakrishnan, P. Periyasamy, S. Pious Missier. *$\hat{\delta}$ -closed sets in ideal topological spaces*, IOSR Journal of mathematics Vol. 11 Issue 1 Ver. II, pp. 01-12, 2015.