Derivations and Right centralizers on 2-torsion free prime rings

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المشتقات والتمركزات اليمينية للحلقات الاولية طليقة الالتواء من النمط

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المستخلص

لتكن R حلقة اولية طليقة الالتواء من النمط 2 ، d مشتقة على R و T تمركز يميني غير صفري على R . في هذا البحث سوف نحقق الابدالية على R المحتوية على مشتقة d وتمركز يميني T محققين واحدة من الخواص التالية لكل x,y∈R:

$T([x, y]) \pm [T(x), T(y)]) = 0$ (1) ،

$T([x, y]) \pm [d(x), d(y)]) = 0$ (2) ،

$T([x, y]) + [d(x), T(y)]) = 0$ (3) ،

$d([x, y]) \pm T([x, y]) = 0$ (4) ،

$d([x, y]) + [d(x), T(y)] = 0$ (5) .

Abstract

Let R be a 2-torsion free prime ring, d be a derivation of R and T be a non zero right centralizer of R. In the present paper, we investigate the commutativity of R admitting derivation d and right centralizer T satisfying any of the following properties, for all x, y ∈R:

(i) $T([x, y]) \pm [T(x), T(y)]) = 0$,

(ii) $T([x, y]) \pm [d(x), d(y)]) = 0$,

(iii) $T([x, y]) + [d(x), T(y)]) = 0$,

(iv) $d([x, y]) \pm T([x, y]) = 0$,

(v) $d([x, y]) + [d(x), T(y)] = 0$.

Keywords: Prime rings, 2-torsion free rings, derivations, right centralizer, commutativity.

Introduction

Let R be a ring. We recall that R is a prime if xRy = (0) implies x=0 or y=0, for all x, y∈R, and a 2-torsion free in case 2x = 0 implies that x = 0 for any x∈ R. An additive mapping d: R → R is called a derivation if d(xy) = d(x)y + xd(y) holds for all x,y ∈ R.

A mapping d is called centralizing if [d(x), x] ∈ Z(R) for all x∈R, in particular, if [d(x), x]= 0 for all x∈R, then it is called commuting of R. We write [x,y] for xy–yx and make extensive use of basic commutator identities

[xy, z]= x[y, z] + [x, z]y , and

[x, yz] = y[x, z] + [x, y]z.

An additive mapping T:R → R is a left (right) centralizer if T(xy)=T(x)y (T(xy)= xT(y)) holds for all x, y ∈ R. A centralizer of R is an additive mapping which is both left and right centralizer. Many authors have investigated the relationship between the commutativity of the classical rings and certain types of derivations. In this direction, the first result was established by Posner (1). Here he proved that if a prime ring R admits a nonzero derivation d such that [d(x), x] ∈Z(R) for all x∈R, then R is commutative. Afterwards, many algebraists refined and extended the Posner's result. In 1991, Borut Zalar worked on centralizers of semi prime rings and prove that Jordan centralizers and centralizers of this rings coincide (2). Joso Vukman, Joso Vukman and Maja Foˇsner developed some remarkable results using centralizers on prime and semi prime rings (3), (4). In this work we investigate the commutativity of R when R is a 2-torsion free prime ring admitting a derivation d and a non-zero right centralizer T, and some properties of them.

Main results

We facilitate our discussion with the following lemma which are necessary for developing the proofs of our theorems:

Lemma 2.1. (Posner.(1)) Let d be a derivation of a prime ring R, and a be an element of R. If a d(x) = 0 for all x ∈R, then either a = 0 or d is zero.

Proof: In a d(x) =0, for all x ∈R, (1)

Replacing x by xy in (1), then

a d(xy)= ad(x)y+axd(y)= 0, for all x,y∈R.(2)

By using (1) in (2), we get

a x d(y) = 0, for all x ,y ∈R.

If d is not zero, that is, if d(y) ≠ 0 for some y ∈R, then, by the definition of a prime ring, a = 0.

Theorem 2.2. Let R be a 2-torsion free prime ring. If R admits a non-zero right centralizer T such that T([x, y]) + [T(x),

$T(y)] = 0$, for all x, y∈ R, then if $T(x)=x$, R is commutative.

Proof: For any x,y∈ R ,we have

$T([x, y]) + [T(x), T(y)] = 0$, for all x,y ∈ R, which gives

$(x+T(x))T(y) – (y+T(y))T(x) =0$,

for all x,y ∈R. (1)

Replacing y by zy in (1), we obtain

$(x+T(x))\, z\, T(y) – z(y+ T(y))T(x) =0$,for all x,y and z∈R (2)

From (1), we have

$(x+T(x))T(y) = (y +T(y))T(x)$ (3)

Substituting (3) in (2) gives

$(x+T(x))zT(y)– z(x+T(x))T(y) =0$,for all x,y and z ∈R. Simplify, we get

$[(x+T(x)), z]\, T(y) =0$, also we get

$[x, z]\, T(y) + [T(x), z]\, T(y) =0$, for all x,y and z ∈R. Since $T(x)=x$, we get

$2\, [x, z]\, T(y) =0$, for all x,z and y ∈R,

since R is 2-torsion free ,we obtain

$[x, z]\, T(y)=0$, for all x,z and y ∈R. (4)

Replacing x by rx in (4), and using (4) again, we get

$[r, z]\, x\, T(y) =0$, for all r,z,x and y ∈ R.

Since R is a prime and $T \neq 0$, we obtain

$[r, z]=0$, for all r and z ∈ R. Thus, R is commutative.

A slight modification in the proof of the above theorem yields the following.

Theorem 2.3. Let R be a prime ring. If R admits a non-zero right centralizer T such that $T([x, y]) = [T(x), T(y)]$, for all x, y∈ R, then if $T(x)=x$, R is commutative.

Theorem 2.4. Let R be a prime ring, and d be a derivation on R. If R admits a non-zero right centralizer T such that $T([x, y]) = [d(x), d(y)]$, for all x,y∈R, then either d=0 or R is commutative.

Proof: For any x,y∈ R , let we have

$T([x, y]) = [d(x), d(y)]$, for all x,y ∈ R,

which gives: $T(xy)– T(yx)- d(x)d(y)+ d(y)d(x) =0$,for all x,y ∈R (1)

Replacing x by yx in (1),we obtain

$yxT(y)- yyT(x)- d(y)xd(y)- yd(x)d(y)+ d(y)d(y)x+d(y)yd(x) =0$, for all x,y ∈R. (2)

From (1), we have

$xT(y)– yT(x)= d(x)d(y)- d(y)d(x)$ (3)

Substituting (3) in (2) gives

$yd(x)d(y)- yd(y)d(x)- d(y)xd(y)- yd(x)d(y)+ d(y)d(y)x+d(y)yd(x) =0$, then we get

$[d(y), y]\, d(x)+ d(y)\, [d(y), x] =0$,

for all x,y ∈R. (4)

Replacing d(y) by x in (4),we obtain

$[x, y]\, d(x) =0$, for all x,y ∈R. (5)

Replacing y by ry in (5), and using (5) again, we get

$[x, r]\, y\, d(x) = 0$, for all x,y and $r \in R$. (6)

Since R is a prime, we obtain either $[x, r]=0$ or $d(x)=0$, for all x,y and $r \in R$. Thus, $d = 0$ or R is commutative.

A slight modification in the proof of the above theorem yields the following.

Theorem 2.5. Let R be a prime ring, and d be a derivation on R. If R admits a non-zero right centralizer T such that

$T([x, y]) + [d(x), d(y)] = 0$, for all $x, y \in R$, then either $d=0$ or R is commutative.

Theorem 2.6. Let R be a 2-torsion free prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that $T([x, y]) + [d(x), T(y)] = 0$, for all x,

$y \in R$, then R is commutative.

Proof: For any $x,y \in R$,we have

$T([x, y]) + [d(x), T(y)] = 0$, for all $x,y \in R$, which gives

$(x+d(x))T(y) - yT(x) + T(y)d(x) = 0$,for all $x,y \in R$ (1)

Replacing y by zy in (1),we obtain

$(x+d(x))\, z\, T(y) - zyT(x) + zT(y)d(x) = 0$,for all x,y and $z \in R$ (2)

From (1), we have

$(x+d(x))T(y) = yT(x) - T(y)d(x)$ (3)

Substituting (3) in (2) gives

$(x+d(x))\, z\, T(y) - z(x+d(x))T(y) = 0$,for all x,y and $z \in R$ (4)

Then, we get $[(x+d(x)), z]\, T(y) = 0$, also we get

$[x, z]\, T(y) + [d(x), z]\, T(y) = 0$, for all x,y and $z \in R$. (5)

Replacing d(x) by x in (5),we obtain

$2\, [x, z]\, T(y)=0$, for all $x,y \in R$.

Since R is 2-torsion free ,we obtain

$[x, z]\, T(y)=0$, for all $x,y \in R$. (6)

Replacing z by rz in (6), and using (6) again, we get $[x, r]\, z\, T(y) = 0$, for all x,y,z and $r \in R$. Since R is a prime and $T \neq 0$, we obtain $[x, r]=0$, for all x and $r \in R$. Thus, R is commutative.

Corollary 2.7. In Theorem 2.6, if R is a prime such that $T([x, y]) = [d(x), T(y)]$, then d is commuting.

Theorem 2.8. Let R be a prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that

$T([x, y]) = [T(x), d(y)]$, for all $x, y \in R$, then R is commutative.

Proof: For any $x,y \in R$,let we have

$T([x, y]) - [T(x), d(y)] = 0$, for all $x,y \in R$, which gives

$xT(y) - T(x)d(y) - (y-d(y))T(x) = 0$,

for all $x,y \in R$ (1)

Replacing x by zx in (1),we obtain

$zxT(y) - zT(x)d(y) - (y-d(y))\, zT(x) = 0$,

fo rall x,y and $\in R$ (2)

Substituting (1) in (2) gives

z(y-d(y))T(x)- (y-d(y)) zT(x) =0 ,for all x,y and z∈R

Then, we get [z, (y-d(y))] T(x) =0,

also we get [z, y] T(x) - [z, d(y)] T(x) =0, for all x,y and z ∈R (3).

Replacing d(y) by z in (3),we obtain

[z, x] T(x) =0, for all x,z ∈R(4).

Replacing z by rz in (4), and using (4) again, we get [r, x] z T(x) =0, for all x,z and r ∈R.

Since R is a prime and T ≠0, we obtain

[r, x]=0, for all x and r ∈ R. Thus, R is commutative.

Corollary 2.9. In Theorem 2.8, if R is a 2-torsion free prime such that T([x, y])+ [T(x), d(y)] =0, then d is commuting.

Theorem 2.10. Let R be a 2-torsion prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that d([x, y]) + [d(x), T(y)]=0, for all x,y∈R, then if T(x) =x for all x∈ R, R is commutative.

Proof: For any x,y∈ R , we have

d([x, y])+[d(x), T(y)] =0, for all x,y ∈ R, which gives

d(x)y+ xd(y)– d(y)x- yd(x) +d(x)T(y) - T(y)d(x)=0 ,for all x,y ∈R ,and also gives

[d(x), y+T(y)] + [x, d(y)] = 0,

for all x,y ∈R (1)

Replacing d(y) by x in (1),we obtain

[d(x), y] + [d(x), T(y)] = 0,

for all x,y ∈R (2)

Replacing d(x) by rd(x) in (2), and using (2) again, we obtain

[r, y] d(x) + [r, T(y)] d(x) = 0, for all x,y and r∈R (3)

Since T(x)=x for all x∈ R, (3) becomes ,

2 [r, y] d(x) = 0, for all x,y and r∈R.

Since R is 2-torsion free, we have

[r,y]d (x) = 0, for all x,y and r∈R (4) .

By using lemma 2.1 in (4), we obtain

[r, y] =0, for all r,y ∈R. Thus, R is commutative.

Theorem 2.11. Let R be a prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that d([x, y]) = T([x, y]), for all x, y∈ R, then, R is commutative.

Proof: For any x,y∈ R , let

d([x, y]) = T([x, y]), which gives

d(xy) - d(yx) - T(xy) + T(yx) =0. Also, gives

x(d(y)-T(y)) - y(d(x)-T(x))+d(x)y- d(y)x= 0, for all x,y ∈ R. (1)

Replacing y by zy in (1), we get

xz(d(y)-T(y))- zy(d(x)- T(x)) + xd(z)y + d(x)zy- d(z)yx - zd(y)x = 0, for all x,y and z∈ R. (2)

Substituting (1) in (2) ,we get

[x, z] (d(y)-T(y)) + [x, d(z)y] + [d(x), z] y =0, for all x,y and z ∈R (3) .

Replacing d(x) by z in (3), we obtain

$[x, z] (d(y)-T(y)) + [x, d(z)y] =0$, for all x,y and z∈R. (4)

Replacing x by zx in (4), we get

$z [x, z] (d(y)-T(y)) + [zx, d(z)y] =0$, for all x,y and z∈R (5) .

Substituting (4) in (5), we get

$d(z)[z,y]x + [z,d(z)] yx = 0$, for all x,y and z∈R. (6)

Replacing z by y, and x by d(x) in (6), we obtain $[y, d(y)] y d(x) =0$, for all x,y ∈R.

Since R is a prime, and d≠0, we obtain

$[y,d(y)] = 0$, for all y∈ R (7) .

Replacing d(y) by rd(y) in (7), and using (7) again, we obtain $[y, r] d(y) =0$, for all r,y∈R, and by using lemma 2.1, we get

$[y, r] =0$, for all r,y ∈R. Thus, R is commutative.

A slight modification in the proof of the above theorem yields the following.

Theorem 2.12. Let R be a prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that

$d([x, y]) + T([x, y]) = 0$, for all x, y∈ R, then R is commutative.

Theorem 2.13. Let R be a prime ring, and d be a non zero derivation on R. If R admits a non-zero right centralizer T such that

$d([x, y]) = [T(x), T(y)]$, for all x, y∈ R, then if T(x)= x, R is commutative.

Proof: For any x,y∈ R , we have

$d([x, y]) = [T(x), T(y)]$, for all x,y ∈ R, which gives

$d(x)y+ xd(y)– d(y)x- yd(x)- T(x)T(y)+ T(y)T(x)=0$,for all x,y ∈R (1)

Replacing y by xy in (1),we obtain

$d(x)xy+ xd(x)y+ xxd(y)– d(x)yx- xd(y)x- xyd(x)- T(x)xT(y)+ xT(y)T(x)=0$,

for all x,y ∈R (2)

Substituting (1) in (2) gives

$[x, T(x)]T(y)+ d(x) [x, y] =0$, for all x,y ∈R. (3)

Since T(x)=x, we obtain $d(x) [x, y] =0$, for all x,y ∈R. (4)

By using lemma 2.1 , we obtain $[x, y] =0$, for all x,y ∈R. Thus, R is commutative.

Theorem 2.14. Let R be a prime ring, and d be a derivation on R.

If R admits a non-zero centralizer T such that $d([x, y]) + [T(x), d(y)]= 0$, for all x, y∈ R, then either d=0 or R is commutative.

Proof: For any x,y∈ R , we have

$d([x, y]) + [T(x), d(y)]= 0$, for all x,y ∈ R, which gives $d(x)y+ xd(y)– d(y)x- yd(x)+ T(x)d(y)- d(y)T(x)=0$, for all x,y ∈R. (1)

Replacing x by yx in (1),we obtain

$d(y)xy- d(y)yx+ yT(x)d(y)- d(y)yT(x)+ y(d(x)y+ xd(y)- d(y)x- yd(x)) =0$,

for all x,y ∈R (2)

Substituting (1) in (2) gives

d(y)[x, y] + [y, d(y)]T(x) =0,
for all x,y ∈R. (3)

Replacing x by xr in (3),we obtain
d(y) x [r, y] + d(y)[x, y] r + [y, d(y)] xT(r) =0, for all x,y and r ∈R. (4)

Substitute (3) in (4) gives
d(y) x [r, y] + [y, d(y)] (xT(r) – T(x) r) =0, for all x,y and r ∈R.

Since T is centralizer, we get
d(y) x [r, y] =0, for all x,y and r ∈R. (5)

Since R is a prime, we obtain either d(y) =0 or [r, y]=0 , for all x,y and r ∈ R. Thus, d =0 or R is commutative.

A slight modification in the proof of the above theorem yields the following.

Theorem 2.15. Let R be a prime ring, and d be a derivation on R. If R admits a non-zero centralizer T such that d([x, y])= [T(x), d(y)] , for all x, y∈ R, then either d=0 or R is commutative.

We now have enough information to prove the following result.

Theorem 2.16. Let R be a 2-torsion free prime ring and d a derivation of R. If R admitting a non-zero right centralizer T satisfying one of the following conditions:

(i) T ([x, y]) + [T(x),T(y)] =0,
(ii) T ([x, y]) = [d(x), d(y)],
(iii) T ([x, y]) = [T(x), d(y)],
(iv) T ([x, y])+[d(x),T(y)] =0,
(v) d ([x, y]) = [T(x), T(y)],
(vi) d ([x, y])+[d(x), T(y)]=0. Then R is commutative or d = 0, for all x,y ∈R.

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